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FINITE-REPETITION THRESHOLD FOR LARGE ALPHABETS

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Abstract. We investigate the Finite-Repetition threshold for 4 and 5-letter alphabets. We show that there exists an infinite Dejean's word on 4 letters (*i.e.* a word without factors of exponent more than $\frac{7}{5}$) containing only two $\frac{7}{5}$-powers. For a 5-letter alphabet, we show that there exists an infinite Dejean's word containing only 60 $\frac{5}{4}$-powers, and we conjecture that this number can be lower down to 45.

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1. Introduction

Following the study of infinite words avoiding repetitions in relation to Dejean's statement on the repetition threshold of alphabets [6] we show that it is possible to impose more constraints on words. We are interested in infinite words whose maximal exponent of its finite factors does not exceed Dejean's threshold and that contain a finite number of factors having the maximal exponent. This introduces the notion of Finite-Repetition threshold (see [2, 3]). Imposing this constraint is not possible on the binary alphabet whose Finite-Repetition threshold is $\frac{7}{3}$ while the Repetition threshold is 2 (see [11, 13]), but can be satisfied for larger alphabets. This confirms the intuition given by the growth rates of words having the smallest exponent according to their alphabet size (see [8, 14]).

Associated with the Finite-Repetition threshold is the smallest number of factors of highest exponent that an infinite word can accommodate (see [1, 7]). We show here that there exists an infinite word on 4 letters containing only 2 $\frac{7}{5}$-powers and no factor of exponent more than $\frac{7}{5}$. The only known proofs of the $\frac{7}{5}$ repetition threshold for 4 letters are due to Pansiot [10] and Rao [12]; their both words contain 24 $\frac{7}{5}$-powers. On 5 letters, the proof of the $\frac{5}{4}$ threshold by Moulin-Ollagnier [9] provides a word with 360 $\frac{5}{4}$-powers of periods 4, 12 and 44. We show that this number can be reduced to 60 and conjecture that it can be lower down to 45, the smallest possible number.

Both results also provide in fact new proofs of the repetition thresholds for the corresponding alphabet sizes, 4 and 5. The same question remains open for larger alphabets.

2. Preliminaries

We denote by Σ_k the set $\{1, 2, ..., k\}$ for $k \geq 2$. A *repetition* in a word w as a pair of words (p, e) where p non empty, and e is a prefix of pe, and pe is a factor of w. The *period* of the repetition is $|p|$, and its *exponent*, E is $\frac{|pe|}{|p|}$. By abuse of notation, we identify sometimes the repetition (p, e) with the factor pe of w. A repetition (p, e) in w over the alphabet Σ_k is a *short repetition* if $|e| < k - 1$, otherwise it is a *kernel repetition*.

A word is *x-free* (resp. x^+*-free*) if it has no repetition of exponent $E \geq x$ (resp $E > x$). A word is called an *E-power* if it is a repetition of exponent E.

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The *repetitive threshold* of order k, denoted $\mathrm{RT}(k)$, is the infimum of maximum exponents of repetition over all infinite words on a k-letter alphabet. The following was conjectured by Dejean [6] and finally proved by several authors (see [5,12]).

$$\mathrm{RT}(k)=\begin{cases}\frac{7}{4} & k=3\\ \frac{7}{5} & k=4\\ \frac{k}{k-1} & k\geq 5 \text{ or } k=2\end{cases}$$

We say that a (infinite) word on k-letters is a *Dejean word* if it is $\mathrm{RT}(k)^+$-free, and a factor is a *limit repetition* if its exponent is $\mathrm{RT}(k)$.

The *finite-repetition threshold* for the alphabet of k letters is the smallest rational number $\mathrm{FRT}(k)$ for which there exists an infinite $\mathrm{FRT}(k)^+$-free word, and containing a finite number of $\mathrm{RT}(k)$-powers. We already know that $\mathrm{FRT}(2)=\frac{7}{3}$ [11,13] and $\mathrm{FRT}(3)=\frac{7}{3}$ [2].

Pansiot proved that the *repetition threshold* for 4-letter alphabet is $\frac{7}{5}$. In order to prove the result, Pansiot used a construction that codes $\frac{k-1}{k-2}$-free word over alphabet Σ_k into a binary word. Let $k\geq 3$ and w be a $\frac{k-1}{k-2}$-free word over Σ_k, of length at least $k-1$. Then every factor of length $k-1$ consist of $k-1$ different letters. The *Pansiot code* of w is the binary word $\mathtt{P}_k(w)$ such that for all $i\in\{1,...,|w|-k+1\}$ (for all $i\geq 1$ if w is infinite):

$$\mathtt{P}_k(w)[i]=\begin{cases}0 & w[i+k-1]=w[i]\\ 1 & w[i+k-1]\notin\{w[i],...,w[i+k-2]\}\end{cases}$$

Note that w is uniquely defined by $P_k(w)$ and $w[1..k-1]$. One can define an inverse operation: for a binary word w, $M_k(w)$ is the word on the alphabet Σ_k such that:

$$\mathtt{M}_k(w)[i]=\begin{cases}i & i<k\\ \mathtt{M}_k(w)[i-k+1] & i\geq k \text{ and } w[i-k+1]=0\\ \alpha & \text{otherwise}\end{cases}$$

where $\{\alpha\}=\Sigma_k\setminus\{\mathtt{M}_k(w)[i-k+1],...,\mathtt{M}_k(w)[i-1]\}$. Note that if $w[i]=i$ for every $i<k$, then $\mathtt{M}_k(P_k(w))=w$.

We shall denote by $\mathbb{S}_k$ the *symmetric group* on k elements, therefore the elements of this set are the permutations of the set $\Sigma_k=\{1,2,...,k\}$. Let $\Psi:\Sigma^*\to\mathbb{S}_k$ be a morphism. A repetition (p,e) is a Ψ*-kernel repetition* if $p\in\ker(\Psi)$.

Let $\varphi:\{0,1\}\to\mathbb{S}_k$ be the morphism such that $\varphi(0)=(1...k-1)$ and $\varphi(1)=(1...k)$. The following Lemma by Moulin-Ollagnier gives a strong relation between kernel repetitions in a word on a k-letter alphabet and φ-kernel repetitions in its Pansiot code.

Lemma 1 ([9]). *Let w be a $\frac{k-1}{k-2}$-free word w on a k-letter alphabet. Then w has a kernel-repetition (p,e) if and only if $P_k(w)$ has a φ-kernel-repetition (p',e') with $|p'|=|p|$, $p'e'=P_k(pe)$ and $|e'|=|e|-k+1$.*

3. Finite-repetition threshold for k=4

Since the *repetition threshold* for 4-letter alphabet is $\frac{7}{5}$, it suffices to show that there exists a $\frac{7}{5}^+$-free infinite word on Σ_4 with finitely many limit repetitions (that is $\frac{7}{5}$-powers). There are two proofs of Dejean's conjecture for $k=4$, by Pansiot [10] and Rao [12]. In both cases the number of limit repetitions contained in the infinite words is 24. This proves that the finite-repetition threshold of 4-letters is $\frac{7}{5}$. In this section, we prove the following:

Theorem 1. *The finite-repetition threshold of 4-letter alphabets is $\frac{7}{5}$ and the minimal number of $\frac{7}{5}$-powers is 2.*

A computer check shows that a word on a 4-letter alphabet for which the maximal exponent of factors is $\frac{7}{5}$ and that contains at most one $\frac{7}{5}$-power has maximal length 230. We give a construction of an infinite

$\frac{7}{5}^+$-free word with only only two $\frac{7}{5}$-powers as a consequence Theorem 1 follows. Let:

$$f : \begin{cases} \mathtt{a} & \to \mathtt{abc} \\ \mathtt{b} & \to \mathtt{cda} \\ \mathtt{c} & \to \mathtt{adc} \\ \mathtt{d} & \to \mathtt{cba} \end{cases}$$

$$g : \begin{cases} \mathtt{a} & \to \mathtt{aacbbaaccbaabcabc} \\ \mathtt{b} & \to \mathtt{aacbacbaabbcaabbc} \\ \mathtt{c} & \to \mathtt{cbaaccbbaccabcabc} \\ \mathtt{d} & \to \mathtt{aacbaccaabbcaabbc} \end{cases}$$

$$h : \begin{cases} \mathtt{a} \to & \mathtt{101101010110110101101101010110101011011010101101101010110101} \\ & \mathtt{011011010101101101010110101011011010101} \\ \mathtt{b} \to & \mathtt{101101010110110101101101010110110101011010101101101010110110} \\ & \mathtt{101011010101101101010110110101011010101} \\ \mathtt{c} \to & \mathtt{101101010110110101101101010110110101011011010101101010110110} \\ & \mathtt{101011011010101101010110110101011011010.} \end{cases}$$

The rest of this section is devoted to the proof of the following theorem.

Theorem 2. *$w_0 = M_4(h(g(f^\infty(\mathtt{a}))))$ is $\frac{7}{5}^+$-free and it contains only two $\frac{7}{5}$-powers:* $(3421432412, 3421)$ *and* $(1423412432, 1423)$*.*

A computer check shows that the Pansiot code of every infinite Dejean word with at most two limit repetitions contains a $h(x)$ as factor, for an $x \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$. Moreover, every Pansiot code of an infinite Dejean word with at most two limit repetitions starting with a $h(x)$ (for $x \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$) must be followed by a $h(y)$, for a $y \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$. Thus the morphism h in our construction is unavoidable, *i.e.* for every Dejean word w which prove Theorem 1, $P_4(w)$ must be the image by h of a ternary word.

Hereafter, we say that a repetition (p, e) is *forbidden* if its exponent is greater than $\mathrm{RT}(k)$, or if it is a limit repetition different from $(3421432412, 3421)$ and $(1423412432, 1423)$. Thus a φ-kernel repetition in a Pansiot code is forbidden if $\frac{|pe|+k-1}{|p|} \geq \mathrm{RT}(k)$. A computer check shows that w_0 has no small forbidden repetition. We show now that $w_1 = h(g(f^\infty(\mathtt{a})))$ has no forbidden φ-kernel repetition. The following properties derive from simple observations:

- f is 3-uniform, g is 17-uniform and h is 99-uniform. Thus $g \circ h$ is 1683-uniform.
- f, g, h and $g \circ h$ are coma-free. (A morphism $f : \Sigma^* \to \Sigma'^*$ is coma-free if whenever $f(xy) = uf(z)v$, then either $u = \epsilon$ or $v = \epsilon$, for every $x, y, z \in \Sigma$ and $u, v \in \Sigma'^*$.)
- The longest common prefix in $\{g \circ h(\mathtt{a}), g \circ h(\mathtt{b}), g \circ h(\mathtt{c}), g \circ h(\mathtt{d})\}$ has size 635 and the longest common suffix has size 990.

The following fact is clarifiable by computer:

Fact 1. *For every $x \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$, $\varphi(h(x)) = (13)$ (*i.e.* the permutation which exchanges* 1 *and* 3*), thus for every $x \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$, $\varphi(h(g(x)) = (13)$ and $\varphi(h(g(f(x))) = (13)$.*

Let $\varphi' : \{0, 1, 2, 3\}^* \to \mathbb{S}_4$ such that $\varphi'(u) = (13)^{|u|}$. Note that $\varphi'(u) = \varphi(h(g(u)) = \varphi(h(g(f(u))))$ since f and g are uniform and of odd size. Thus (p, q) is a φ'-kernel repetition if (p, q) is a repetition, and $|p|$ is even. Applying Lemma 1, we conclude:

Corollary 1. *Let (p_0, e_0) be a repetition in w_0. If $|e_0| \geq 3$, then $w_1 = h(g(f^\infty(\mathtt{a})))$ has a φ-kernel-repetition (p_1, e_1), with $|e_1| = |e_0| - 3$.*

Lemma 2. *Let (p_1, e_1) be a φ-kernel-repetition of $w_1 = h(g(f^\infty(\mathtt{a})))$. If $|e_1| \geq 3365$, then $w_2 = f^\infty(\mathtt{a})$ has a φ'-kernel-repetition (p_2, e_2) with $|e_2| \geq \left\lceil \frac{|e_1|-1625}{1683} \right\rceil$ and $|p_1| = 1683 \cdot |p_2|$.*

Proof. Suppose w.l.o.g. that (p_1, e_1) is a maximal repetition, *i.e.* there is an occurrence of p_1e_1 in w_1 which cannot be extended to the left or to the right without loosing the property of being a repetition with the same period. If $|e_1| \geq 3365$, either $g \circ h(\mathtt{a})$, $g \circ h(\mathtt{b})$ or $g \circ h(\mathtt{c})$ appears as a factor in e_1. Since $g \circ h$ is coma-free and 1683-uniform, $|p_1|$ is a multiple of 1683. Let $|p_1| = 1683 \times k$. Then there is a factor $u = a_1 \dots a_l$ in w_2 such that $g \circ h(u) = vp_1e_1v'$, v is a proper prefix of $g \circ h(a_1)$ and v' is a proper suffix of a_l. Since (p_1, e_1) is a repetition of size $k \times 1683$, for every $k < i < l$, $a_i = a_{i-k}$. Since p_1e_1 is maximal on the left, if $|v| < 693$, then $a_1 = a_k$, and since p_1e_1 is maximal on the right, if $|v'| < 1048$, then $a_l = a_{l-k}$. If $a_1 \neq a_k$ and $a_l \neq a_{l-k}$, then $(a_2 \dots a_k, a_{k+1} \dots a_{l-1})$ is a repetition of w_2 of period k, and $l - k - 1 \geq \left\lceil \frac{|e_1|-1625}{1683} \right\rceil$. If $a_1 = a_k$ and $a_l \neq a_{l-k}$, then $(a_1 \dots a_{k-1}, a_k \dots a_{l-1})$ is a repetition of w_2 of period k, and $l - k \geq \left\lceil \frac{|e_1|-635}{1683} \right\rceil$. If $a_1 \neq a_k$ and $a_l = a_{l-k}$, then $(a_2 \dots a_k, a_{k+1} \dots a_l)$ is a repetition of w_2 of period k, and $l - k \geq \left\lceil \frac{|e_1|-990}{1683} \right\rceil$. If $a_1 = a_k$ and $a_l = a_{l-k}$, then $(a_1 \dots a_{k-1}, a_k \dots a_l)$ is a repetition of w_2 of period k, and $l - k + 1 \geq \left\lceil \frac{|e_1|}{1683} \right\rceil$. In all cases, w_2 has a repetition (p_2, e_2) of period $|p_2| = k = \frac{|p_1|}{1683}$ and with $|e_2| \geq \left\lceil \frac{|e_1|-1625}{1683} \right\rceil$. Moreover, since $\varphi'(p_2) = \varphi(p_1)$ and $\varphi(p_1) = \mathrm{Id}$, (p_2, e_2) is a φ'-kernel repetition of w_2. □

The proof of the following Lemma is similar, and is omitted.

Lemma 3. *If (p_2, e_2) is a φ'-kernel repetition of $w_2 = f^\infty(\mathtt{a})$ with $|e_2| \geq 5$, then w_2 has a φ'-kernel-repetition (p'_2, e'_2) with $|e'_2| \geq \left\lceil \frac{|e_2|-2}{3} \right\rceil$ and $|p_2| = 3 \cdot |p'_2|$.*

Lemma 4. *Suppose that w_2 has a φ'-kernel-repetition (p_2, e_2) with $|e_2| \geq 5$ and $\frac{|e_2|+1}{|p_2|} \geq \frac{2}{5}$. Then there exists a φ'-kernel-repetition (p'_2, e'_2) with $|p_2| = 3 \cdot |p'_2|$ and $\frac{|e'_2|+1}{|p'_2|} \geq \frac{2}{5}$.*

Proof. By Lemma 3,

$$\frac{2}{5} \leq \frac{|e_2| + 1}{|p_2|} \leq \frac{3 \cdot |e'_2| + 3}{3 \cdot |p'_2|} = \frac{|e'_2| + 1}{|p'_2|}.$$

□

The following fact can be verified by computer check:

Fact 2. *There is no φ'-kernel-repetition (p_2, e_2) with $2 \leq |e_2| < 5$ and $\frac{|e_2|+1}{|p_2|} \geq \frac{2}{5}$ in w_2.*

Thus by Lemma 4:

Corollary 2. *There is no φ'-kernel-repetition (p_2, e_2) with $2 \leq |e_2|$ and $\frac{|e_2|+1}{|p_2|} \geq \frac{2}{5}$ in w_2.*

Lemma 5. *w_1 has no φ-kernel-repetition (p_1, e_1) with $|e_1| \geq 3 \cdot 1683$ and $\frac{|e_1|}{|p_1|} \geq \frac{2}{5}$.*

Proof. Suppose that w_1 has a φ-kernel-repetition (p_1, e_1) with $|e_1| \geq 3 \cdot 1683$ and $\frac{|e_1|}{|p_1|} \geq \frac{2}{5}$. By Lemma 2, w_2 has a φ'-kernel repetition (p_2, e_2) with $|e_2| \geq 2$ and

$$\frac{2}{5} \leq \frac{|e_1|}{|p_1|} \leq \frac{1683 \cdot |e_2| + 1625}{1683 \cdot |p_2|} < \frac{|e_2| + 1}{|p_2|}.$$

By Corollary 2, w_2 has no such φ'-kernel repetition. Contradiction. □

To show that w_0 has no forbidden kernel repetition, it suffice to show that w_1 has no forbidden φ-kernel repetition (p_1, e_1) with $|p_1| \leq 12622$, which has been verified by a computer check.

4. Finite-repetition threshold for k=5

Moulin-Ollagnier gave a proof of Dejean's conjecture for $k = 5$ [9]:

$$m : \begin{cases} \mathtt{0} & \rightarrow \mathtt{010101101101010110110} \\ \mathtt{1} & \rightarrow \mathtt{101010101101101101101} \end{cases}$$

then $M_5(m^\infty(\mathtt{0}))$ is $\frac{5}{4}^+$-free. However it contains 360 of such powers, of which a third have period 4, a third period 12 and the remaining period 44. This proves that the finite-repetition threshold of 5-letter alphabets is $\frac{5}{4}$.

This section is devoted to the minimum number of limit repetitions in a Dejean word on 5-letters. We give a Dejean word with only 60 limit repetitions, and we conjecture that the minimal number of limit repetitions in a Dejean word is 45. Similarly to the proof of Theorem 1, here we are looking for a morphic word w_1 such that $w_0 = M_5(w_1)$ has the desired property. This can be done by the following construction:

$$f : \begin{cases} \mathtt{a} & \rightarrow \mathtt{aaabbababbaaabbaabb} \\ \mathtt{b} & \rightarrow \mathtt{aabbbaababaabbbaabb} \end{cases}$$

$$g : \begin{cases} \mathtt{a} & \rightarrow \mathtt{aaaababbbbababaaaababbb} \\ \mathtt{b} & \rightarrow \mathtt{bbbbabaaaabababbbbabaaa} \end{cases}$$

$$h : \begin{cases} \mathtt{a} \rightarrow & \mathtt{110110101010110110101010110110101011011010101101101101010110} \\ & \mathtt{11011011010101011011010101101101010110110110101010110} \\ \mathtt{b} \rightarrow & \mathtt{110110101011011010101101101010101101101101101010110110110101} \\ & \mathtt{0110110101011011010101011011010101011011011010101010110}. \end{cases}$$

Theorem 3. *$w_0 = M_5(h(g(f^\infty(\mathtt{a}))))$ is $\frac{5}{4}^+$-free and it contains only 60 of $\frac{5}{4}$-powers, all of which have period 4.*

The following properties will help the proof of the Theorem 3:

- f is 19-uniform, g is 29-uniform and h is 113-uniform. Thus $g \circ h$ is 3277-uniform.
- f, g, h and $g \circ h$ are coma-free.
- The longest common prefix in $\{g \circ h(\mathtt{a}), g \circ h(\mathtt{b})\}$ has size 11 and the longest common suffix has size 24.
- For every $x \in \{\mathtt{a}, \mathtt{b}\}$, $\varphi(h(x)) = (12)(354)$, thus for every $x \in \{\mathtt{a}, \mathtt{b}\}$, $\varphi(h(g(x)) = (12)(345)$ and $\varphi(h(g(f(x))) = (12)(345)$.

Let $\varphi' : \{0,1,2,3,4\}^* \rightarrow \mathbb{S}_5$ such that $\varphi'(u) = [(12)(345)]^{|u|}$. Thus (p, q) is a φ'-kernel repetition if and only if (p, q) is a repetition, and $|p|$ is divisible by 6.

Lemma 6. *Let (p_1, e_1) be a φ-kernel-repetition of $w_1 = h(g(f^\infty(\mathtt{a})))$. If $|e_1| \geq 6553$, then $w_2 = f^\infty(\mathtt{a})$ has a φ'-kernel-repetition (p_2, e_2) with $|e_2| \geq \left\lceil \frac{|e_1|-34}{3277} \right\rceil$ and $|p_1| = 3277 \cdot |p_2|$.*

Lemma 7. *If $|e_2| \geq 37$, then $w_2 = f^\infty(\mathtt{a})$ has a φ'-kernel-repetition (p'_2, e'_2) with $|e'_2| \geq \left\lceil \frac{|e_2|-8}{19} \right\rceil$ and $|p_2| = 19 \cdot |p'_2|$.*

Here, we adapt the same approach as the Section 3 (Lemma 4 and Fact 2) with cooperating the size of the morphism f and the exponent $\frac{5}{4}$, the next Corollary follows:

Corollary 3. *There is no φ'-kernel-repetition (p_2, e_2) with $6 \leq |e_2|$ and $\frac{|e_2|+1}{|p_2|} \geq \frac{1}{4}$ in w_2.*

Lemma 8. *w_1 has no φ-kernel-repetition (p_1, e_1) with $|e_1| \geq 6 \cdot 3277$ and $\frac{|e_1|}{|p_1|} \geq \frac{1}{4}$.*

The proof is a direct consequence of Lemma 6 and 7, therefore it is sufficient to check that w_0 has no forbidden repetition (p_0, e_0) with $|p_0| \leq (6 \cdot 3277 \cdot 4) = 78648$. This claim can be verified by a basic computation which also reveals that there are only 60 limit repetitions (p_0, e_0) in w_0, and for every limit repetition, $|e_0| = 1$.

5. Conjecture on the minimum number of limit repetitions when k=5

The following facts have been verified by computer check.

Fact 3.

- *A Dejean word on a* 5*-letter alphabet that contains at most* 44 *limit repetitions has size at most* 4648*.*
- *A Dejean word on a* 5*-letter alphabet that contains at most* 45 *limit repetitions, and such that every limit repetition has period* 4*, has size at most* 7330*.*

Also based on computer checks, we conjecture the followings:

Conjecture 1.

- *There exists an infinite Dejean word on a* 5*-letter alphabet with only* 45 *limit repetitions.*
- *There exists an infinite Dejean word on a* 5*-letter alphabet with only* 46 *limit repetitions, and such that every limit repetition has period* 4*.*

6. Finite-repetition threshold for k>5 and existing morphisms

Looking at the existing proofs for Dejean's conjecture shows in fact $\mathrm{FRT}(k) = \mathrm{RT}(k)$ for $k \geq 6$, that is, the known constructions of Dejean's words for $k \geq 5$ have only finitely many limit repetitions.

- $6 \leq k \leq 11$ (cases are by Moulin-Ollagnier [9]), and $12 \leq k \leq 38$ (cases are by Rao [12]). In both sets of proofs, authors show that if the Pasiot's code of the constructed word w contains a φ-kernel repetition (p, e) with e markable, then the word has a φ-kernel repetition of smaller period (p', e') with $\frac{|e|}{|p|} \leq \frac{|e'|}{|p'|}$ ([9, Section 3.5], [12, Corollary 9]). By Lemma 1, (p, e) (resp. (p', e')) corresponds to a kernel repetition of period $|p|$ and size $|pe| + k - 1$ in w (resp. $|p'|$ and size $|p'e'| + k - 1$). Since $\frac{|pe|+k-1}{|p|} < \frac{|p'e'|+k-1}{|p'|} \leq RT(t)$, (p', e') does not correspond to a limit repetition. Thus w cannot have arbitrary long limit kernel repetitions, and we have $\mathrm{FRT}(k) = \mathrm{RT}(k)$. Moreover, a simple computer check reveals that in each of these cases, all limit repetitions have period $k - 1$, and thus there are at most $k!$ of limit repetitions.
- $k > 38$. These cases were proved by Carpi. A close inspection of [4, Proposition 8.2] shows this proposition remains valid if the factor is a long enough limit repetition. Thus Carpi's construction cannot have arbitrary long limit repetitions, therefore we have $\mathrm{FRT}(k) = \mathrm{RT}(k)$.

We conclude by two straightforward open questions.

- Is it possible to construct Dejean's words such that the only allowed limit repetitions have period $k - 1$, for every $k > 38$? Maybe a closer inspection of Carpi's construction will give the result.
- Let $\mathrm{LR}(k)$, $k \geq 3$, be the minimum number of limit repetitions in a Dejean's word on k-letters. We know that $\mathrm{LR}(3) = 2$ [2], $\mathrm{LR}(4) = 2$ and $45 \leq \mathrm{LR}(5) \leq 60$. Can we find a lower or an upper bound for $\mathrm{LR}(k)$?

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